Effects of asymmetric stiffness on parametric instabilities of rotor

Chandan Kumar and Vikas Rastogi[*]

Department of Mechanical Engineering, Sant Longowal Institute of Engineering and Technology, Longowal-148106, Punjab, India.

Abstract

This work deals with effects of asymmetric stiffness on the dynamic behaviour of the rotor system. The analysis is presented through an extended Lagrangian Hamiltonian mechanics on the asymmetric rotor system, where symmetries are broken in terms of the rotor stiffness. The complete dynamics of asymmetries of rotor system is investigated with a case study. In this work, a mathematical model is developed considering symmetry breaking of a finite rotor due to stiffness. The natural frequency and amplitude of the rotor are obtained analytically through extended Lagrangian formulation. The asymmetries in rotor are also modeled through bond graph modeling technique for the computational analysis. The simulation result shows a considerable agreement with the analytical results. The limiting dynamics of rotor is shown and analyzed through simulation.

1. Introduction

Rotor is generally considered as one of the most important components in large rotating machineries such as generators, power plants turbines and aero industries. The presence of any type of defects (such as cracks, notches, slits, asymmetries) in any structure and machine may lead to catastrophic failures. So, there is a great need to study such flaws or defects so that necessary corrective or preventive actions may be taken. The initiation of fatigue cracks or other defects in structures or machines or their elements mainly causes a reduction in stiffness, which generally changes the dynamic characteristics of the machines. A detailed study of the vibrational behavior of a cracked

rotating shaft is an important and interesting problem for engineers working in the area of rotor dynamics. The rotating shaft has a discontinuous stiffness value, so the system is piecewise linear, exhibiting non-smoothness in the vector field. A great deal of research effort has been devoted by several researchers to the study of non-linear response of asymmetric rotor and its variation with system parameters.

Among the various analytical techniques and methods used by various researchers, Lagrangian mechanics is one of the areas, where several works have been reported in literature. It is a well known phenomenon that an asymmetric rotating component produces non potential and dissipative forces, and the classical Lagrange's equation cannot analyze

---

[*]*Corresponding author*
Email address: rastogivikas@yahoo.com

the dynamics of such system with non holonomic constraints, non-potential forces, dissipative forces, gyroscopic forces and general class of systems with time fluctuating parameters. As such Lagrangian cannot be worked out due to non-conservative forces involved in the system. So some additional information of system interior and exterior is needed in generating extended Lagrangian equations [1- 3], which may be applicable to an asymmetric rotor system.

In classical mechanics, the greatest advantage of Lagrangian formulation is that it brings out connection between conservation laws and important symmetry properties of dynamical systems. However, it suffers a lot due to its inherent limitations in presence of time fluctuating parameters, general dissipation and gyroscopic coupling. Several authors made attempts to enhance the Lagrange's equation in their several ways to incorporate all these influences of dissipation, constraints of non-conservative dynamical system and show the connection between symmetries and existence of conserved quantities, and thus finding invariants of motion of dynamical systems. Some significant works in this area were reported by Vujanovic [4, 5] and Djukic [6, 7] in their papers. Vujanovic [4, 5] has formulated a method for finding the conserved quantities of non-conservative holonomic system, which is based on the differential variational principle of D'Alembert.

Knowledge of conservation laws (First integrals invariants, constant of motion) is of great importance in the study of dynamical systems such as rotors. One of the most interesting aspects of conservation laws is their relationship to invariance properties [8, 9] of dynamical systems. The symmetries of the system also provide useful information to derive the constant of motion of the system as they reduce the complexity of the dynamical systems such as cracked rotor. Noether's theorem [10] played a significant role in determination of invariants of motion. Noether's theorem allows exploitation of symmetries of the system to arrive at invariant of motion. Extended Noether's theorem with Umbra's Hamiltonian provides the useful

insight of dynamics of the asymmetric rotor systems.

To overcome all these limitations and enlarging the scope of Lagrangian-Hamiltonian mechanics, a new proposal of additional time like variable 'Umbra-time' was made by Mukherjee [11] and this new concept of Umbra-time leads to a peculiar form of equation, which is termed as Umbra-Lagrange's equation. A brief and candid commentary on idea of Umbra Lagrangian is given in various references [12, 13]. This idea was further consolidated by presenting an important issue of invariants of motion for the general class of system by extending Noether's theorem [14]. This notion of Umbra-time is again used to propose a new concept of Umbra-Hamiltonian, which is used along with the extended Noether's theorem to provide an insight into the dynamics of systems with symmetries. One of the most important insights gained from the Umbra-Lagrangian formalism is that its underlying variational principle [15] is possible, which is based on their cursive minimization of functionals. Once such a least action principle is established, many significant results of analytical mechanics may be extended to a general class of system. The Umbra-Lagrangian theory has been used successfully to study invariants of motion for non-conservative mechanical and thermo-mechanical systems [16] and also for continuous systems [17].

Recently Mukherjee et al [18] applied Umbra-Lagrangian to study dynamics of an electro-mechanical system comprising of an induction motor driving an elastic rotor. The basic aim of this research work is mainly focused on extending the Lagrangian-Hamiltonian mechanics for asymmetric rotor system, where symmetries are broken in terms of asymmetries in stiffness, mass or in damping. This is in line to broaden the scope of Umbra-Lagrangian formulation applied to this rotor dynamic research.

In this paper, symmetry of the rotor is broken in terms of stiffness only, which may further be applied in similar cases also.

2. Methodology

In several papers, Mukherjee et al. [15- 18] introduced a concise and modified form of Lagrange's equation. They manifested the use of this new scheme to arrive at system models in the presence of time fluctuating parameters, general dissipation and gyroscopic couplings etc. In this scheme, real and virtual energies (or work) are separated by introduction of an additional time like parameter, which is termed as 'Umbra-time'. The prefix 'Umbra' was appended to all type of energies, and corresponding Lagrangian was termed as the "Umbra Lagrangian". The basic idea presented in leading to Umbra-Lagrangian and Umbra-Lagrange's equation has been briefly expressed in Ref. [15].

The broad principle on which the creation of Umbra-Lagrangian and other relevant energies are based can be summarized in Ref. [15] and Umbra-Lagrange's equation [15, 16] for a general class of systems may be written as:

$$\frac{d}{dt}\left\{ \underset{\eta \to t}{Lim} \frac{\partial L^*}{\partial \dot{q}_i(\eta)}\right\} - \underset{\eta \to t}{Lim} \frac{\partial L^*}{\partial q_i(\eta)} = 0,$$

for $i = 1 \ldots n$.

$$(1)$$

The Umbra-Hamiltonian [23] may be represented as

$$H^*\left[\boldsymbol{q}(\eta), \boldsymbol{p}(\eta), \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t\right] = \dot{\boldsymbol{q}}(\eta)\,\boldsymbol{p}(\eta)$$
$$- L^*\left[\boldsymbol{q}(\eta), \dot{\boldsymbol{q}}(\eta), \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t\right]$$

$$(2)$$

The Umbra-Hamiltonian H^* is composed of two components as H_i^* and H_e^*. H_i^* is the interior Hamiltonian, which does not depend on any function of real displacement, real velocity and real time, and H_e^* is the rest of the Umbra-Hamiltonian, called the exterior Hamiltonian, thus:

$$H^* = H_i^*\left\{q(\eta), p(\eta)\right\} + H_e^*\left\{q(\eta), p(\eta), q(t), \dot{q}(t), t\right\}.$$

$$(3)$$

The two theorems of the Umbra-Hamiltonian [15] may be given as

$$\underset{\eta \to t}{Lim}\left[\frac{dH^*}{d\eta}\right] = 0$$

(Theorem 1)

$$\frac{dH_i^*}{dt} = -\underset{\eta \to t}{Lim}\left[\frac{dH_e^*}{d\eta}\right]$$

(Theorem 2)

Corollary of Theorem 2

If for a system $\underset{\eta \to t}{Lim}\left[\dfrac{dH_e^*}{d\eta}\right] = 0$,

then $H_i^*\big(q(t), p(t)\big)$ is a constant of motion.

Noether's theorem [10] states that, if the Lagrangian of a system is invariant under a family of single parameter groups, then each such group renders a constant of motion. The extended Noether's theorem, as discussed in papers [15, 16], may lead to a constant of motion, or trajectories, on which some dynamical quantity remains conserved.

The Umbra-Lagrangian may be defined on extended manifold, which consists of real displacements and velocities as well as Umbra-displacements and velocities and real time [15], i.e.

$$L^* = L^*\big(t, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), \boldsymbol{q}(\eta), \dot{\boldsymbol{q}}(\eta)\big)$$

Here, the super dot ($\cdot$) denotes a derivative with respect to real time or Umbra time, depending on the context. Unlike the classical formulation, this analysis requires single but extended manifold comprising of both Umbra and real displacements and velocities and real time. The Umbra-Lagrangian of a system admits several one-parameter transformation groups, and then the infinitesimal generator [19, 20] corresponding to j^{th} parameter (or group) may be decomposed as follows:

$$\boldsymbol{V}^j = \boldsymbol{V}_\eta^j + \boldsymbol{V}_t^j, \text{where } j = 1 \ldots m \qquad (4)$$

The general forms for $\boldsymbol{V}_\eta^j$ and $\boldsymbol{V}_t^j$ would be:

$$\boldsymbol{V}_\eta^j = \sum_{i=1}^n \alpha^{j,i}\, \frac{\partial}{\partial q_i(\eta)} + \sum_{i=1}^n \beta^{j,i}\, \frac{\partial}{\partial \dot{q}_i(\eta)},$$

$$(5)$$

and $\mathbf{V}_t^j = \sum\limits_{i=1}^{n} \gamma^{j,i} \dfrac{\partial}{\partial q_i(t)} + \sum\limits_{i=1}^{n} \xi^{j,i} \dfrac{\partial}{\partial \dot{q}_i(t)}$,

$$(6)$$

where $\alpha^{j,i}$, $\beta^{j,i} = \dfrac{d\alpha^{j,i}}{d\eta}, \gamma^{j,i}$ and $\xi^{j,i} = \dfrac{d\gamma^{j,i}}{dt}$

are the general functions of Umbra and real displacement and real time. The fact that the given the Umbra-Lagrangian is invariant under the jth transformation may then be expressed as:

$$\left\{ \mathbf{V}^j\left(L^*\right) \right\} = 0$$

$$(7)$$

Using Eq. (1) along-with Eq. (5) and (6) in the previous Eq. (7), the extended Noether's theorem may be obtained and written as

$$\dfrac{d}{dt} \left\{ \lim_{\eta \to t} \sum\limits_{i=1}^{n} \dfrac{\partial L^*}{\partial \dot{q}_i(\eta)} \mathbf{V}_\eta^j\left(q_i(\eta)\right) \right\} = - \lim_{\eta \to t} \left\{ \mathbf{V}_t^j\left(L^*\right) \right\},$$

with $j = 1...m$.

$$(8)$$

In terms of the differential one-forms dq_i and dL^*, the above relation may be expressed as:

$$\dfrac{d}{dt} \left\{ \lim_{\eta \to t} \left(\sum\limits_{i=1}^{n} \dfrac{\partial L^*}{\partial \dot{q}_i(\eta)} dq_i(\eta)\left(\mathbf{V}_\eta^j\right) \right) \right\} =$$
$$- \lim_{\eta \to t} \left\{ dL^*\left(\mathbf{V}_t^j\right) \right\},$$

$$(9)$$

The term on the left-hand side is the classical Noether term while the term on the right-hand side is additional and termed here as modulatory convection term, which is useful to find the insight of the dynamics of the rotor-motor system.

3. Development of model with asymmetries using extended Lagrangian-Hamiltonian mechanics

Let us consider a rotor with internal and external damping and transverse stiffness due to flexure of the shaft driven by a dc motor [21] as shown in Fig. 1. Small asymmetries in stiffness are also considered in the model. As mass of the shaft carrying the disk is ignored when compared with that of the disk, one may

approximate the rotor by two discrete pairs of equivalent springs and dashpots in the two orthogonal transverse directions. Such dampers include external damping and a part of the internal damping. The additional circulatory forces due to the internal damping are introduced using modulated gyrator elements in the bond graph model. The twisting of the shaft is assumed to be negligible on account of high torsional stiffness. However, more refined model including the twisting effect may be considered as this extension will retain SO (2) symmetry of the system and may be analyzed in a similar manner. As no unbalance on the disk is assumed, at steady state, the torsional dynamics will be completely redeemed on account of torsional damping. Hence in this analysis, the torsional dynamics have been ignored.
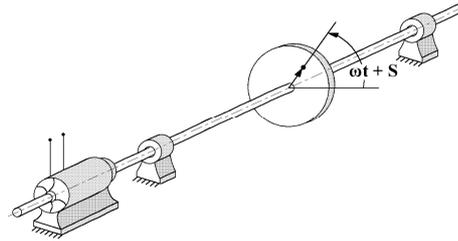
Fig. 1. Rotor with DC motor having asymmetries in stiffness.

The bond graph model of the system in Fig. 1 is shown in Fig. 2 with usual causal orientation suitable for derivation of system equations and simulation (for details see Ref. [18]). The flow activated C element appended to bond 2 (C_2) is only an observer to record the total rotation of the shaft and the rate of rotation, otherwise has no influence on the dynamics of the system.

3.1 The Umbra-Lagrangian for asymmetric rotor system

The Umbra-Lagrangian for the system is obtained following the procedure outlined in ref [15]. The Umbra-Lagrangian corresponding to the bond graph model shown in Fig. 2 is

$$L^* = \frac{1}{2}\left\{ \begin{array}{c} m\{\dot{x}^2(\eta) + \dot{y}^2(\eta)^2\} \\ -\left[Kx^2(\eta) + (K-\Delta K)y^2(\eta) + J\dot{\theta}^2(\eta)\right] \\ -\left[x(\eta)\, y(\eta)\right]\begin{bmatrix} 0 & R_i\dot{\theta}(t) \\ -R_i\dot{\theta}(t) & 0 \end{bmatrix}\begin{Bmatrix} x(t) \\ y(t) \end{Bmatrix} \end{array} \right\}$$
$$-(R_a+R_i)(x(\eta)\dot{x}(t)+y(\eta)\dot{y}(t))-$$
$$\{(R_i(x(t)\dot{y}(t)-y(t)\dot{x}(t))-R_b(\dot{\theta}(t)-\Omega)\}\theta(\eta$$

$$(10)$$

In Eq. (10), m is the mass of the rotor, K is the stiffness of the shaft, ΔK is the asymmetry in stiffness. J is the moment of inertia of the rotor mass, R_i and R_a are the internal and external damping of the rotor, x and y are the displacements in real or Umbra time, θ is the angular displacements in η or t times, and R_b is the resistance of bearings.

Applying the extended Noether's theorem, one may obtain the Noether's rate equation as

$$\frac{d}{dt}\left[\begin{array}{c} m\,(\dot{x}(t)\,y(t) - \dot{y}(t)\,x(t) \\ -\Delta K\,(y(t)\,x(t) \end{array}\right]$$
$$= -\dot{\theta}(t)\,R_i\,(x^2(t) + y^2(t))$$
$$+ (R_a + R_i)(x(t)\,\dot{y}(t)$$
$$- \dot{x}(t)\,y(t))$$

$$(11)$$

One may assume an orbit of
$x(t) = A\cos\omega t, \; y(t) = A\sin\omega t$,
to obtain:

$$-\dot{\theta}(t)R_iA^2 = -A^2\omega(R_a+R_i)$$
$$\dot{\theta}(t) = \frac{\omega(R_a+R_i)}{R_i}$$
So $\dot{\theta}(t) = \omega(1+\frac{R_a}{R_i})$ (12)

The Noether's rate equation comes out to be

$$\frac{d}{dt}\left[m(\dot{x}(t)\,y(t) - \dot{y}(t)x(t) - \Delta K(y(t)x(t))\right] = 0$$

$$(13)$$

$$\leftarrow --- Symmetric- \rightarrow\leftarrow Unsymmetric- \rightarrow$$

The Eq. (13) may have two important terms, if unsymmetric part is put to be zero, i.e. there is no asymmetry in stiffness, one may obtain invariants of motions as:

$$\frac{d}{dt}\left[m(\dot{x}(t)\,y(t) - \dot{y}(t)x(t))\right] = 0 \quad (14)$$

3.2 The Umbra-Hamiltonian for asymmetric rotor system

Considering Eq. (3), the Umbra-Hamiltonian of the system may be written as

$$H^* = \frac{1}{2}\left\{ \begin{array}{c} \left(\dfrac{p_x^2(\eta)}{m_{xx}} + \dfrac{p_y^2(\eta)}{m_{yy}}\right) \\ + Kx^2(\eta) + (K-\Delta K)y^2(\eta) + \dfrac{1}{J}p_\theta^2(\eta) \end{array}\right\}$$
$$+ \left[x(\eta)\quad y(\eta)\right]\begin{bmatrix} 0 & R_i\dot{\theta}(t) \\ -R_i\dot{\theta}(t) & 0 \end{bmatrix}\begin{bmatrix} x(\eta) \\ y(\eta) \end{bmatrix}\{H_i^*\}$$
$$+ (R_a+R_i)(\dot{x}(t)x(\eta) + \dot{y}(t)y(\eta))$$
$$+ \left\{ \begin{array}{c} R_i(x(t)\dot{y}(t) - y(t)\dot{x}(t)) \\ + R_b(\dot{\theta}(t) - \Omega) \end{array}\right\}\theta(\eta)\{H_e^*\}$$

$$(15)$$

The Eq. (15) has interior Hamiltonian as $\{H_i^*\}$ and exterior Hamiltonian as $\{H_e^*\}$.

Applying the theorem 2 of Umbra-Hamiltonian, one may obtain:

$$\underset{\eta \to t}{Lim}\left[\frac{dH_e^*}{d\eta}\right] = R_i\dot{\theta}(t)\left[\begin{array}{c} \dot{x}(\eta)y(t) \\ -\dot{y}(t)x(t) \end{array}\right]$$
$$+ (R_a+R_i)(\dot{x}^2(t) + \dot{y}^2(t))$$
$$+ \left\{ \begin{array}{c} R_i(x(t)\dot{y}(t) - y(t)\dot{x}(t)) \\ + R_b(\dot{\theta}(t) - \Omega) \end{array}\right\}\dot{\theta}(t)$$

Fig. 2. Bond graph model of rotor with asymmetries in stiffness with DC motor and artificial flow sources to obtain Umbra-Lagrangian.

Assuming once again an orbit

$$x(t) = A\cos\omega t \quad \text{and} \quad y(t) = A\sin\omega t,$$

one may obtain

$$\underset{\eta\to t}{Lim}\left[\frac{d H_e^*}{d\eta}\right] = -R_i\dot{\theta}(t)A^2\omega$$

$$+ (R_a + R_i)A^2\omega^2 \qquad\qquad \{A\}$$

$$+ \left\{R_iA^2\omega + R_b\left(\dot{\theta}(t) - \Omega\right)\right\}\dot{\theta}(t) \qquad \{B\}$$

Setting {A} equal to zero yields

$$\dot{\theta}(t) = \omega\left(1 + \frac{R_a}{R_i}\right)$$

and setting {B} equate to zero gives the amplitude of rotor

$$A = \sqrt{\frac{R_b}{R_i\omega}\left[\Omega - \omega\left(1 + \frac{R_a}{R_i}\right)\right]} \qquad (16)$$

when:

$$\underset{\eta\to t}{Lim}\left[\frac{d H_i^*}{d\eta}\right] = 0,$$

one obtains the following expression

$$\frac{d}{dt}\left\{\begin{array}{l}\left(\dfrac{p_x^2(\eta)}{m_{xx}} + \dfrac{p_y^2(\eta)}{m_{yy}}\right) + Kx^2(\eta) \\[2mm] + (K - \Delta K)y^2(\eta) + \dfrac{1}{J}p_\theta^2(\eta)\end{array}\right\} = 0 \qquad (17)$$

Now putting

$$p_x(t) = m_{xx}\dot{x}(t), \ p_y(t) = m_{yy}\dot{y}(t) \text{ and}$$

$$p_\theta(t) = J\dot{\theta}(t)$$

in Eq. (17), one may obtain the equation as

$$(m\ddot{x} + Kx)\dot{x} + (m\ddot{y} + Ky - \Delta Ky)\dot{y} = 0$$

$$(m\ddot{x} + Kx)\dot{x} + \{m\ddot{y} + (K - \Delta K)y\}\dot{y} = 0 \qquad (18)$$

The two types of frequencies are available in Eq. (18). Adopting the method as provided in Ref [22], one may obtain the natural frequency in *x-x* direction if symmetry part is there, one may have

$$\omega_{xx} = \sqrt{\frac{K}{m}} \qquad (19)$$

If unsymmetric part is there, one obtain the natural frequency in *y-y* direction

$$\omega_{yy} = \sqrt{\frac{K - \Delta K}{m}} \qquad (20)$$

the final natural frequency, may be represented as:

$$\omega^2 = \frac{\omega^2_{xx} + \omega^2_{yy}}{2}$$

or:

$$\omega = \sqrt{\frac{K - \frac{\Delta K}{2}}{m}} \qquad (21)$$

The result obtained in Eq. (21) will be validated through simulation in later subsections of the paper.

4. Simulation study

The main purpose of this study is to gain an insight into the dynamical behaviour and visualizing the electromechanical interactions of asymmetric rotor. Moreover, it provides validation of the theoretical results obtained above. The bond graph model of a motor with asymmetric rotor with internal and external damping was simulated on the software SYMBOLS-Shakti [23, 24], with parameters shown in Table 1. In order to show effectiveness of the analysis described above, a
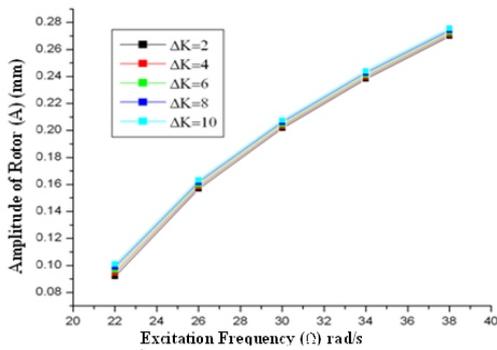
Table 1. Simulation parameters.

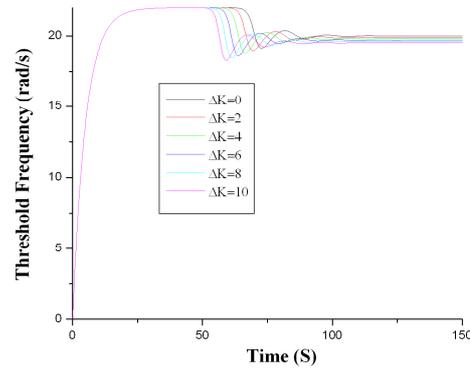| Parameter	Value
K	100 N/m
m	1 Kg
J	1 Kg-m^2
R_i	5 N-s/m
R_a	5 N-s/m
R_b	0.2 N-s/m
Ω	22 rad/s
ΔK	2-10 N/m
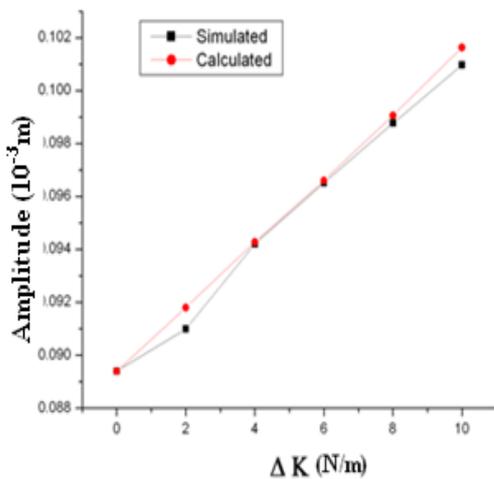
Fig. 3. Amplitude of the rotor Vs Asymmetries in, stiffness.
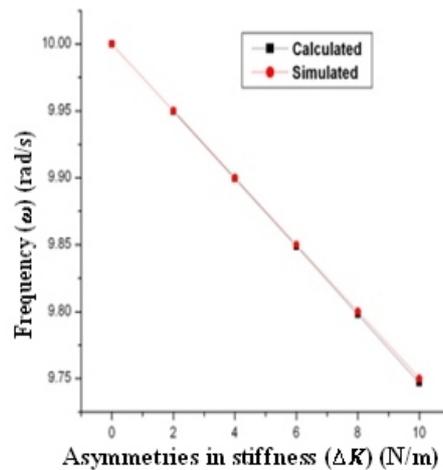
Fig. 4. Frequency of the Rotor Vs Asymmetries in stiffness.

Fig. 5. Amplitude of the rotor Vs excitation frequency with different stiffness variation.

Fig. 6. Threshold frequency Vs time with different stiffness variation.

series of parameter variation studies are performed. The dynamic behaviour of such rotor-motor system is governed by the rotor mass, shaft stiffness, internal/external damping, excitation frequency and bearing resistance. The system is extremely sensitive to all these parametric variations. The parameters for simulation of the model are given in Table 1.

Using the parameters given in Table 1, the threshold speed of instability becomes twice the natural frequency of the shaft.

It can be seen from the plot of Fig. 3 that the amplitude of rotor at different stiffness variation confirms to the analytical results obtained in this paper. There is a marginal difference in simulated and calculated value of rotor amplitude at different stiffness variation. A similar behaviour is also observed in Fig. 4, where calculated frequency exactly matches with the simulated frequency obtained. A general trend of reduction in the natural frequencies at various stiffness variations is observed.

The plot of Fig. 5 shows the simulated value of rotor amplitudes at various excitation frequencies with stiffness variation. The amplitude of the rotor increases with increase in stiffness variation. Good agreement between analytical and numerical simulation results can be stated. Here again the vibrational behavior

of the analytical rotor can be approximated very well by the numerical simulation.

The following diagram presents some more insight of the numerical simulation. It is apparent from the Fig. 6 that the threshold frequency of the rotor decreases with increase in ΔK and conforms to the angular speed obtained through Eq. (12). This is basically the frequency at which the asymmetric rotor is spinning in actual. The phenomenon is of great use for monitoring the cracks in rotor system.

5. Conclusions

The asymmetries of the rotor in terms of stiffness variation were analyzed through an extended Noether's theorem, to obtain invariants of motion of asymmetric rotor system. The limiting behaviour has been observed for an electro-mechanical system, consisting of an asymmetric flexible shaft that is having asymmetries in stiffness only. Similar asymmetries in mass and damping may also be attempted by this method, which presents a novel method to obtain frequencies and amplitude analytically. The stability issues may also be attempted by this procedure. The Umbra-Lagrangian for such a system was generated from its bond graph model. Analysis has showed that the Umbra-Lagrangian for such

asymmetric system remained invariant and along with Umbra-Hamiltonian, lead to some significant aspects of the limiting dynamics of this complex asymmetric rotor-motor system. The results obtained theoretically have been verified through simulation results.

Acknowledgement

References

[1] D. C. Karnopp, R. C. Rosenberg, and D. L. Margolis, *System Dynamics: A Unified Approach*, John-Wiley and Sons Inc., USA, (1990).

[2] P. C. Breedveld, and G. Dauphin-Tanguy, *Bond Graphs for Engineers*, North-Holland, Amsterdam, (1992).

[3] P. Gawthrop, and L. Smith, *MetaModeling: Bond graphs and Dynamic Systems*, Prentice-Hall, (1996).

[4] B. Vujanovic, "Conservation Laws of Dynamical Systems via d'Alembert's Principle", *Inter. J. Non-Linear Mech.*, Vol. 13, pp. 185, (1978).

[5] B. Vujanovic, and S. E. Jones, *Variational Methods in Non-Conservative Phenomena*, Academic press, Boston, (1989).

[6] Di. Djukic, "A Procedure for Finding First Integrals of Mechanical Systems with Gauge-Variant Lagrangians", *Inter. J. Non-Linear Mech.*, Vol. 8, pp. 479, (1973).

[7] Dj. Djukic, and B. Vujanovic, "Noether Theory in Classical Mechanics", *Acta Mech.*, Vol. 23, pp. 17, (1975).

[8] D. C. Karnopp, "Lagrange's Equations for Complex Bond Graph Systems", *Journal of the Dynamic Systems, Measurement, and Control*, ASME Trans., Vol. 99, pp. 300-306, (1977).

[9] A. Mukherjee, And A. K. Samantaray, "Umbra-Lagrange's Equations through Bond Graphs", *Proceedings of International Conference on Bond Graph Modeling and Simulation* (ICBGM), Phoenix, Arizona, USA, pp. 168-174, (1997).

[10] E. Noether, "Invariant Variationsprobleme", *Ges. Wiss. Worthington*, Vol. 2, pp. 235, (1918).

[11] A. Mukherjee, "Junction Structures of Bond graph Theory from Analytical Mechanics Viewpoint", Proc. of *CISS-1st Conference of International Simulation Societies*. Zurich, Switzerland, pp. 661-666, (1994).

[12] F. T. Brown, *Engineering System Dynamics*, 2nd, CRC, Taylor and Francis, (2007).

[13] A. Mukherjee, "The Issue of Invariants of Motion for General Class of Symmetric Systems through Bond Graph and Umbra-Lagrangian", *Proc. of Int. Conf. of Bond Graph Modelling and Simulation ICBGM'01*, Phoenix, Arizona, USA, pp. 295, (2001).

[14] A. Mukherjee, V. Rastogi, and A. Das Gupta, "A Methodology for finding Invariants of Motion for Asymmetric Systems with Gauge-transformed Umbra Lagrangian Generated by Bond Graphs", *Simulation*, Vol. 82, No. 4, pp. 207-226, (2006).

[15] A. Mukherjee, V. Rastogi, and A. Dasgupta, "Revisiting Umbra-Lagrangian-Hamiltonian Mechanics: Its Variational foundation and extension of Noether's theorem and Poincaré-Cartan Integral", *International Journal of Non-Linear Mechanics*, Vol. 46, No. 5, pp. 745-757, (2011).

[16] V. Rastogi, A. Mukherjee, "Conservation laws for gauge-variant Umbra-Lagrangian in Classical mechanics using Bond graphs", *Simulation: Transactions of Society of Modeling and Simulation International*, Vol. 87, No. 4, pp. 301-312, (2011).

[17] A. Mukherjee, V. Rastogi, A. Dasgupta, "Extension of Lagrangian-Hamiltonian Mechanics for Continuous Systems – Investigations of Dynamic Internally

Damped Rotor Driven through Dissipative Coupling", *Nonlinear Dynamics: Journal of Springer Verlag*, Vol. 58, No. 1-2, pp. 107-127, (2009).

[18] A. Mukherjee, V. Rastogi, and A. Das Gupta, "A Study of a Bi-Symmetric Electro mechanical System through Umbra Lagrangian Generated by Bond Graphs and Noether's Theorem", *Simulation*, Vol. 83, No. 9, pp. 611-630, (2007).

[19] S. Hassani, *Mathematical Physics*, Springer-Verlag, New York, (1999).

[20] J. V. Jose, and E. J. Saletan, *Classical Dynamics- A Contemporary Approach*, Cambridge University Press, Cambridge, (1998).

[21] A. Mukherjee, and R. Karmakar, R., *Modelling and Simulation of Engineering System through Bond Graph*, Narosa Publishing House, New Delhi, Reprinted by CRC press for North America and by Alpha Science For Europe,(2000).

[22] J. M. Krodkiewski, *Dynamics of Rotors*, The University of Melbourne, Australia, (2006).

[23] URL://www.Symbols2000.com, (2000).

[24] A. Mukherjee, and A. K. Samantaray, *SYMBOLS-Shakti User's Manual*, High-Tech Consultants, STEP Indian Institute of Technology, Kharagpur, India, (2006).