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A new positive definite semi-discrete mixed finite element solution for parabolic equations

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Abstract

In this paper, a positive definite semi-discrete mixed finite element method was presented for two-dimensional parabolic equations. In the new positive definite systems, the gradient equation and flux equations were separated from their scalar unknown equations. Also, the existence and uniqueness of the semi-discrete mixed finite element solutions were proven. Error estimates were also obtained for the semi-discrete schemes. Finally, a numerical example was presented to show theoretical results.

1. Introduction

A parabolic partial differential equation models the flow of water in a porous medium, which pore spaces may contain water and air. Recently, many researchers have studied numerical methods for parabolic equations, such as finite element methods [1–4], mixed finite element methods [5–9], finite volume element method [10], etc.

Chen [11–13] proposed a new mixed method and proved some mathematical theories for second-order linear elliptic equations. Yang et al. [14–16] proposed a new mixed finite element method (called the splitting positive definite mixed finite element procedure) for treating the pressure equation of parabolic type in a nonlinear parabolic system which described a model for compressible flow displacement in a porous medium. Compared with the standard mixed finite elements methods with quite difficult numerical solutions because of losing positive definite properties, the proposed one did lead to any saddle point equations.

In this paper, the following parabolic equations were considered:

$$p_t - \nabla\cdot(K(x,t)\nabla p) = f(x,t), (x,t)\in \Omega\times J,$$
$$p(x,t) = 0, (x,t)\in \partial\Omega\times J,$$

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$$p(x,0) = p_0(x), x \in \Omega, \quad (1.1)$$

where Ω is a bounded convex domain in R^2 with the smooth boundary of $\partial\Omega$, $J = (0,T]$.

At first, the following assumptions were made for the parabolic equation. The positive constants K_* and K^* existed such that

$$0 < K_* \leq K(x,t) \leq K^*.$$

By $L^p(\Omega)$ denoting the standard Banach space, the standard Sobolev space with norm $\|\cdot\|_{m,p}$ can be denoted by $W^{m,p}(\Omega)$.

The functional spaces $W = L^2(\Omega)$,

$V = H(div;\Omega) = \{v \in (L^2(\Omega))^2; \nabla \cdot v \in L^2(\Omega)\}$

can be introduced.

Also, the auxiliary variables

$$\tau = -\nabla p, \qquad \sigma = K(x,t)\tau$$

can be introduced.

Then, the equivalent system of parabolic equations can be derived for Eq. (1.1)

$$p_t + \nabla \cdot \sigma = f(x,t), (x,t) \in \Omega \times J,$$

$$\tau + \nabla p = 0, (x,t) \in \Omega \times J, \quad (1.2)$$

$$\sigma - K(x,t)\tau = 0, (x,t) \in \Omega \times J,$$

with the initial values

$$p(x,t) = p_0(x),$$

$$\tau(x,0) = -\nabla p_0(x),$$

$$\sigma(x,0) = K(x,0)\tau(x,0).$$

Then, the following weal formulation of Eq. (1.2) can be given by

$$(p_t, w) + (\nabla \cdot \sigma, w) = (f, w), \forall w \in W,$$

$$(\tau, v) - (p, \nabla \cdot v) = 0, \forall v \in V,$$

$$(\sigma, v) - (K\tau, v) = 0, \forall v \in V. \quad (1.3)$$

The plan of this paper is as follows. In Section 2, a new semi-discrete mixed finite element scheme is constructed for parabolic equations. In Section 3, an error estimate is derived for the mixed finite element solutions. Finally, a numerical example is presented in Section 4 to show theoretical results. A conclusion is presented in Section 5.

2. A new semi-discrete mixed finite element scheme

From the second item of Eq. (1.3), the following can be derived:

$$(\tau_t, v) - (p_t, \nabla \cdot v) = 0, \forall v \in V. \quad (2.1)$$

By taking $w = \nabla \cdot v$ in the first item of Eq. (1.3) for $v \in V$ and then substituting it into (2.1), a new equivalent weak formulation of Eq. (1.3) was obtained

$$(p_t, w) + (\nabla \cdot \sigma, w) = (f, w), \forall w \in W,$$

$$(\tau_t, \nabla \cdot v) + (\nabla \cdot \sigma, \nabla \cdot v) = (f, \nabla \cdot v), \forall v \in V,$$

$$(\sigma, v) - (K\tau, v) = 0, \forall v \in V. \quad (2.2)$$

T_h denotes a quasi-uniform partition of Ω into rectangles or triangles with the partition step h. Also, W_h and V_h are Brezzi -Douglas-Fortin-Marini mixed finite element spaces on the partition T_h.

Now, the positive definite semi-discrete mixed finite element method for Eq. (2.2) can consist of determining

$$(p_h, \tau_h, \sigma_h) \in W_h \times V_h \times V_h$$

such that

$$(p_{ht}, w_h) + (\nabla \cdot \sigma_h, w_h) = (f, w_h), \forall w_h \in W_h,$$

$$(\tau_{ht}, \nabla \cdot v_h) + (\nabla \cdot \sigma_h, \nabla \cdot v_h) = (f, \nabla \cdot v_h), \forall v_h \in V_h,$$

$$(\sigma_h, v_h) - (K\tau_h, v_h) = 0, \forall v_h \in V_h, \quad (2.3)$$

with a given initial approximation

$$(p_h^0, \tau_h^0, \sigma_h^0) \in W_h \times V_h \times V_h.$$

Theorem 1. There exists a unique discrete solution to systems (2.3).

Proof. Let $\{\psi_i(x)\}_{i=1}^{N_1}$ and $\{\varphi_i(x)\}_{i=1}^{N_2}$ be bases of W_h and V_h, respectively. Let

$$p_h = \sum_{i=1}^{N_1} p_i(t)\psi_i(t),$$

$$\tau_h = \sum_{i=1}^{N_2} \tau_j(t)\varphi_j(t),$$

$$\sigma_h = \sum_{i=1}^{N_2} \sigma_j(t)\varphi_j(t),$$

and substitute these formula into Eq. (2.3) while choosing $w_h = \psi_m$, $v_h = \varphi_l$. Eq. (2.3) can be written in a vector matrix form as: find $\{ \mathrm{P}(t), \Gamma(t), \Sigma(t) \}$ such that $\forall t \in (0,T]$

$$D\mathrm{P}'(t) + E\Sigma(t) = G(t),$$
$$A\Gamma'(t) + B\Sigma(t) = F(t),$$
$$A\Sigma(t) - C\Gamma(t) = 0,$$

where

$$A = \left((\varphi_j, \varphi_l)\right)_{N_2\times N_2},$$
$$B = \left((\nabla\cdot\varphi_j, \nabla\cdot\varphi_l)\right)_{N_2\times N_2},$$
$$C = \left((K\varphi_j, \varphi_l)\right)_{N_2\times N_2},$$
$$D = \left((\psi_i, \psi_m)\right)_{N_1\times N_1},$$
$$E = \left((\nabla\cdot\varphi_j, \psi_m)\right)_{N_1\times N_2},$$
$$F = \left((f(t), \nabla\cdot\varphi_l)\right)^T_{1\times N_2},$$
$$G = \left((f(t), \varphi_m)\right)^T_{1\times N_1},$$
$$\mathrm{P} = \left(p_1(t), p_2(t), \cdots, p_{N_1}\right)^T,$$
$$\Gamma = \left(\tau_1(t), \tau_2(t), \cdots, \tau_{N_2}\right)^T,$$
$$\Sigma = \left(\sigma_1(t), \sigma_2(t), \cdots, \sigma_{N_2}\right)^T.$$

It is easy to see that D and A are symmetric positive definite. Using the theory of differential equations, this system had a unique solution.

3. Error estimates

Now, an operator R_h is defined from V onto V_h in mixed finite element spaces such that

$$(\nabla\cdot(\sigma - R_h\sigma), v_h) = 0, \forall v_h \in \nabla\cdot V_h,$$
$$\| \sigma - R_h\sigma \|_{(L^2(\Omega))^2} \le Ch^{k+1} \| \sigma \|_{(W^{k+1,2}(\Omega))^2},$$
$$\| \nabla\cdot(\sigma - R_h\sigma) \|_{L^2(\Omega)} \le Ch^{k+1} \| \nabla\cdot\sigma \|_{W^{k+1,2}(\Omega)}$$

Also, the L^2-project operator P_h is defined from $L^2(\Omega)$ onto V_h such that

$$(v - P_h v, v_h) = 0, \forall v \in L^2(\Omega), v_h \in V_h,$$
$$\| v - P_h v \|_{L^2(\Omega)} \le Ch^{k+1} \| v \|_{H^{k+1,2}(\Omega)}, \forall v \in H^{k+1}(\Omega)$$

Using the definitions of the operators R_h and P_h, t he following theorem can be easily derived.

Theorem 2. Assume that the solution of Eq. (2.2) has regular properties that

$$p_t, p_{tt} \in L^2(H^{k+1}(\Omega)),$$
$$\tau_t, \tau_{tt}, \sigma_t, \sigma_{tt} \in L^2(H^{k+1}(\Omega))$$

Then, the following estimates are assumed

$$\| (\tau - R_h\tau)_t \|_{(L^2(\Omega))^2} \le Ch^{k+1} \| \tau_t \|_{(W^{k+1,2}(\Omega))^2},$$
$$\| (\tau - R_h\tau)_{tt} \|_{(L^2(\Omega))^2} \le Ch^{k+1} \| \tau_{tt} \|_{(W^{k+1,2}(\Omega))^2},$$
$$\| (\sigma - R_h\sigma)_t \|_{(L^2(\Omega))^2} \le Ch^{k+1} \| \sigma_t \|_{(W^{k+1,2}(\Omega))^2},$$
$$\| (p - P_h p)_t \|_{L^2(\Omega)} \le Ch^{k+1} \| p_t \|_{W^{k+1,2}(\Omega)}$$

Let

$$p - p_h = p - P_h p + P_h p - p_h,$$
$$\tau - \tau_h = \tau - R_h\tau + R_h\tau - \tau_h,$$
$$\sigma - \sigma_h = \sigma - R_h\sigma + R_h\sigma - \sigma_h$$

Subtracting (2.3) from (2.2) results in obtaining

$$((p_h - p)_t, w_h) + (\nabla\cdot(\sigma_h - \sigma), w_h) = 0, \forall w_h \in W_h,$$
$$((\tau_h - \tau)_t, \nabla\cdot v_h) + (\nabla\cdot(\sigma_h - \sigma), \nabla\cdot v_h) = 0, \forall v_h \in V_h,$$
$$((\sigma_h - \sigma), v_h) - (K(\tau_h - \tau), v_h) = 0, \forall v_h \in V_h.$$

Using the operators R_h and P_h, the following is derived

$$((p_h - P_h p)_t, w_h) + (\nabla\cdot(\sigma_h - R_h\sigma), w_h)$$
$$= ((p - P_h p)_t, w_h),$$

$$((\tau_h - R_h\tau)_t, \nabla\cdot v_h) + (\nabla\cdot(\sigma_h - R_h\sigma), \nabla\cdot v_h)$$
$$= ((R_h\tau - \tau_h)_t, \nabla\cdot v_h) + (\nabla\cdot(\sigma - R_h\sigma), \nabla\cdot v_h),$$
$$((\sigma_h - R_h\sigma), v_h) - (K(\tau_h - R_h\tau), v_h)$$
$$= ((R_h\sigma - \sigma), v_h). + (K(R_h\tau - \tau), v_h)$$

Combination of Theorem 2, Cauchy-Schwarz's inequality and Gronwall's lemma causes to easily obtain the following theorem.

Theorem 3. Assume that the solution of Eq. (2.2) has regular properties that

$p_t, p_{tt} \in L^2(H^{k+1}(\Omega))$,

$\tau_t, \tau_{tt}, \sigma_t, \sigma_{tt} \in L^2(H^{k+1}(\Omega))$.

Then, the following error estimates are presented

$$\| p - p_h \|_{L^\infty(L^2(\Omega))} \le Ch^{k+1},$$
$$\| \tau - \tau_h \|_{L^\infty(L^2(\Omega))} + \| \sigma - \sigma_h \|_{L^\infty(L^2(\Omega))} \le Ch^{k+1},$$
$$\| \nabla\cdot\sigma - \nabla\cdot\sigma_h \|_{L^\infty(L^2(\Omega))} \le Ch^{k+1}.$$

4. Numerical example

In the section, an example is provided to validate the theoretical results. The considered equation is

$$p_t - \nabla\cdot(K(x,t)\nabla p) = f(x,t), (x,t) \in \Omega\times(0,1],$$
$$p(x,t) = 0, (x,t) \in \partial\Omega\times(0,1],$$
$$p(x,0) = 0, x \in \Omega,$$

where

$$\Omega = [0,1]^2, \quad K(x,t) = p(x,t), \quad x = (x_1, x_2).$$

Let the exact solution of the above equations be

$$p(x,t) = x_1 x_2 (1 - x_1)(1 - x_2)\sin t.$$

Then, the right term can be obtained

$$\begin{aligned} f(x,t) &= p_t - \nabla\cdot(p\nabla p) \\ &= x_1 x_2 (1 - x_1)(1 - x_2)\cos t \\ &\quad - (1 - 6x_1 - 6x_1^2) x_2^2 (1 - x_2)^2 \sin^2 t \\ &\quad - (1 - 6x_2 - 6x_2^2) x_1^2 (1 - x_1)^2 \sin^2 t \end{aligned}$$

In order to show efficiency of the new positive definite mixed finite element methods, the methods were compared with the standard mixed finite element methods.

The numerical results are shown in Tables 1 and 2.

From the numerical results of Tables 1 and 2, it can be shown that the test results coincided with the theoretical analysis and the new positive definite mixed finite element methods spent less time than the standard mixed finite element methods. Then, it would be clear that the new positive definite mixed finite element methods are more efficient.

Table 1. $L^\infty(L^2)$ errors and CPU time of the standard mixed finite element methods.

h	$L^\infty(L^2)$ errors	CPU time (s)
$\frac{1}{4}$	5.67245e-03	126.08
$\frac{1}{8}$	2.81856e-03	655.61
$\frac{1}{16}$	1.27335e-03	3868.13

Table 2. $L^\infty(L^2)$ errors and CPU time of the new positive definite mixed finite element methods.

h	$L^\infty(L^2)$ errors	CPU time (s)
$\frac{1}{4}$	5.68956e-03	2.4246
$\frac{1}{8}$	2.90786e-03	11.7073
$\frac{1}{16}$	1.36892e-03	63.4119

5. Conclusions

In this paper, a positive definite semi-discrete mixed finite element method was analyzed for parabolic equations. In the new positive definite systems, the gradient equation and flux equations were separated from their scalar unknown equations. The existence and uniqueness were proven for the semi-discrete mixed finite element solutions. Next, error estimates were derived for the semi-discrete schemes. Finally, a numerical example was provided to prove the efficiency of the new positive definite mixed finite element methods. In Fig. 1, a flow chart of this paper is presented in order to show the main process.

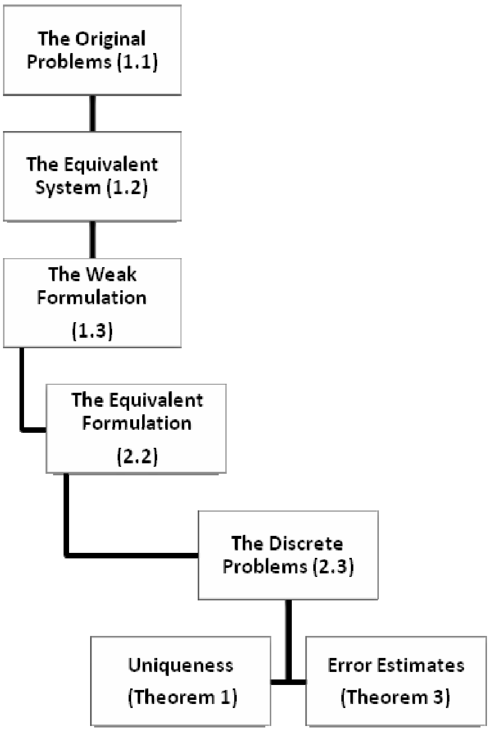

Fig. 1. The method flow chart.

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