Water hammer simulation by explicit central finite difference methods in staggered grids

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Abstract
Four explicit finite difference schemes, including Lax-Friedrichs, Nessyahu-Tadmor, Lax-Wendroff and Lax-Wendroff with a nonlinear filter are applied to solve water hammer equations. The schemes solve the equations in a reservoir-pipe-valve with an instantaneous and gradual closure of the valve boundary. The computational results are compared with those of the method of characteristics (MOC), and with the results of Godunov's scheme to verify the proposed numerical solution. The computations reveal that the proposed Lax-Friedrichs and Nessyahu-Tadmor schemes can predict the discontinuities in fluid pressure with an acceptable order of accuracy in cases of instantaneous and gradual closure. However, Lax-Wendroff and Lax-Wendroff with nonlinear filter schemes fail to predict the pressure discontinuities in instantaneous closure. The independency of time and space steps in these schemes are allowed to set different spatial grid size with a unique time step, thus increasing the efficiency with respect to the conventional MOC. In these schemes, no Riemann problems are solved; hence field-by-field decompositions are avoided. As provided in the results, this leads to reduced run times compared to the Godunov scheme.

1. Introduction

Transient flow in piping systems is generally caused by changes in flow conditions due to rapid closing or opening of valves, or due to start up or shutdown of pumps. Other causes of transient flow are load rejection of turbines, seismic excitation and pipe rupture. The phenomenon is generally called pressure surge or water hammer. Water hammer involves large transient pressure variations which can cause major problems such as noise, vibration, pipe collapse, etc.

In order to prevent damage, water hammer can be suppressed and controlled by devices like surge tanks, air chambers, flexible hoses, pump flywheels, relief valves, and rupture disks. In practice, water hammer analyses are carried out to judge whether these quite expensive devices are necessary and, if so, what their dimensions should be [1]. Predicted maximum pressures determine the required strength of the pipework. Kwon and Lee simulated transient flow in a pipe involving backflow preventers using both

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experimental and three different numerical models of the method of characteristics model (MOC), the axisymmetrical model and the implicit scheme model. The results of different computer models agree well with the experimental data [2]. Afshar and Rohani applied an implicit MOC to a problem of transient flow caused by the failure of a pump with and without a check valve and compared the results with those of the explicit MOC. Results showed that the implicit MOC can be used for any combination of devices to accurately predict the variations of head and flow in the pipeline system [3]. Sabbagh-Yazdi et al. applied a second-order explicit Godunov-type scheme to water hammer problems. The minimum and maximum of the computed pressure waves were in close agreement with the analytical solution and laboratory data [4]. However, the method still fails in the precise prediction of discontinuities. Zhao and Ghidaoui applied first- and second-order Godunov-type schemes for water hammer problems. Numerical tests showed that the first-order Godunov gives the same results to the MOC with space-line interpolation [5]. Chaudhry and Hussaini solved water hammer equations by three explicit finite-difference schemes (MacCormack's method, Lambda scheme and Gabutti scheme). Their study revealed that for the same accuracy, second-order schemes required fewer computational nodes and less computer time as compared to those required by the first-order MOC [6]. Tijsseling and Bergant proposed a method based on the MOC, but a numerical grid is not required. The water hammer equations without friction have been solved exactly for the time-dependent boundary and constant (steady state) initial conditions with this method. Their method was the simplicity of the algorithm (recursion) and the fast and accurate (exact) calculation of transient events but calculation time strongly increased the events of longer duration [7]. Hou et al. [8] simulated water hammer with the corrective smoothed particle method (CSPM). The CSPM results are in good agreement with conventional MOC solutions.

This paper aims at the investigation of four explicit finite difference solutions of water hammer and their comparison with the largely established solutions of MOC and Godunov. The implemented finite difference methods are fast, accurate and simple to program. The four schemes are two-step variant of the Lax-Friedrichs (LxF) method, the Nessyahu-Tadmor based (NT) method, two-step variant of the Lax Wendroff (LxW) method, and the LxW method with a nonlinear filter. A reservoir-pipe-valve system, with both sudden and gradual valve-closure patterns, is taken into account to assess the results. To this end, Matlab codes based on the explicit central finite difference methods are provided. The computational results of these methods, as well as those of MOC and Godunov's scheme, are provided and discussed in detail.

2. Mathematical modelling

2.1. Governing equation

If neglecting friction terms, the following continuity and momentum equations govern one-dimensional transient lfow in pipes [9, 10]:

$$\frac{\partial V}{\partial x} + \frac{g}{c^2} \frac{\partial H}{\partial t} = 0 \qquad (1)$$

$$\frac{\partial V}{\partial t} + g \frac{\partial H}{\partial x} = 0 \qquad (2)$$

where V = fluid velocity, H = fluid pressure head, g = gravitational acceleration, c = wave velocity, x = coordinate axis along the conduit length and t = time.

Equations (1-2) are simplified unsteady pipe flow equations in which convective transport terms are neglected. A full derivation of these equations can be found in many water hammer texts e.g., [9-11]

2.2. Solution procedures

The computational grids consist of individual cells with spatial grid size Δx and time step Δt (see Fig. 1.). In this study, multi-step methods are used to enhance convergence and accuracy. Multi-step methods, which use finite difference relations at split time levels, work well

especially when they are applied to non-linear hyperbolic equations [12].

The two governing equations can be written as the following compact form:

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{\partial \mathbf{f(u)}}{\partial x} = 0, \ \mathbf{u} = \binom{H}{V}, \qquad (3)$$

$$A = \begin{pmatrix} 0 & -\frac{c^2}{g} \\ -g & 0 \end{pmatrix}, \mathbf{f(u)} = A\mathbf{u} = \begin{pmatrix} -\frac{c^2}{g}V \\ -gH \end{pmatrix}$$
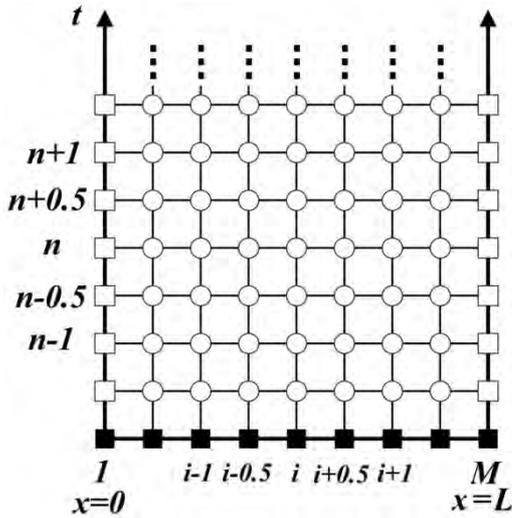

Fig. 1. Stencil for two time steps method.

2.3. Lax-Friedrichs method

The LxF method is the finite difference based numerical method appropriate for the solution of hyperbolic PDEs. This method is actually a prototype of many central schemes. The LxF method is conservative and monotone; therefore, this is a total variation diminishing (TVD) method. Like the original Godunov method, the LxF scheme is based on a piecewise constant approximation of the solution, but it does not require solving a Riemann problem for time advancing and only uses flux estimates [13]. The stability condition is the $\frac{c\Delta t}{\Delta x} \leq 1$, when c is the corresponding wave speed and Δt and Δx are time step and spatial step, respectively.

In this method, firstly, a half time step is taken based on LxF scheme on a staggered mesh.

Next, the second half step is implemented based on LxF to arrive at the solution on the original mesh. This discretization procedure, which adapts the two-step variant of LxF method on the staggered grid, is applied on Eq. (3). Accordingly, the first half step becomes [14]:

$$\frac{\mathbf{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}} - \frac{1}{2}(\mathbf{u}_{i+1}^n + \mathbf{u}_i^n)}{\frac{\Delta t}{2}} = \frac{\mathbf{f}(\mathbf{u}_{i+1}^n) - \mathbf{f}(\mathbf{u}_i^n)}{2\frac{\Delta x}{2}} \ \rightarrow \qquad (4)$$

$$\mathbf{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{u}_{i+1}^n + \mathbf{u}_i^n) + \frac{\Delta t}{2}\left(\frac{\mathbf{f}(\mathbf{u}_{i+1}^n) - \mathbf{f}(\mathbf{u}_i^n)}{\Delta x}\right)$$

Note that this step has to be applied for all spatial nodes in the time level $n + \frac{1}{2}$.

In the second half step, the desired unknowns are obtained as follows:

$$\frac{\mathbf{u}_i^{n+1} - \frac{1}{2}(\mathbf{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}} + \mathbf{u}_{i-\frac{1}{2}}^{n+\frac{1}{2}})}{\frac{\Delta t}{2}} = \frac{\mathbf{f}(\mathbf{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}}) - \mathbf{f}(\mathbf{u}_{i-\frac{1}{2}}^{n+\frac{1}{2}})}{2\frac{\Delta x}{2}} \ \rightarrow \qquad (5)$$

$$\mathbf{u}_i^{n+1} = \frac{1}{2}\left(\mathbf{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}} + \mathbf{u}_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right)$$

$$+ \frac{\Delta t}{2}\left(\frac{\mathbf{f}(\mathbf{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}}) - \mathbf{f}(\mathbf{u}_{i-\frac{1}{2}}^{n+\frac{1}{2}})}{\Delta x}\right)$$

In Fig. 2, the stencil of the two-step LxF scheme discussed before is plotted.

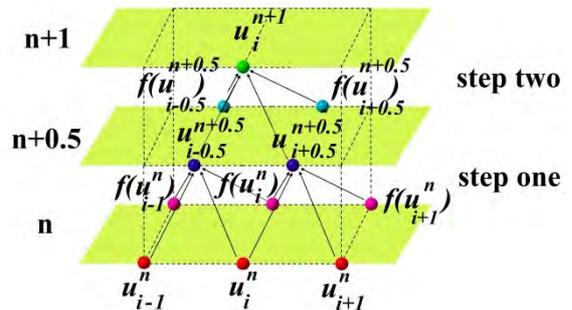**Fig. 2.** The stencil of the two-time step LxF.

2.4. Nessyahu-Tadmor method

Actually, the prototype of NT method is LxF scheme. The stability condition again is $\frac{c\Delta t}{\Delta x} \leq$ 1. It is based on a staggered grid and uses the

reconstruction of Monotonic Upstream-Centered (MUSCL) type piecewise linear interpolants in space, oscillation-suppressing nonlinear limiters, and the midpoint quadrature rule for the numerical integration with respect to time [13]. On this basis, the discretized form of Eq. (3) with NT method on the staggered grid reads:

$$\mathbf{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{u}_i^n + \mathbf{u}_{i+1}^n) + \frac{1}{8}(d\mathbf{u}_i - d\mathbf{u}_{i+1}) \quad (6)$$
$$+\lambda_{half}\left[\mathbf{f}\left(\mathbf{u}_{i+1}^{n+\frac{1}{4}}\right) - \mathbf{f}\left(\mathbf{u}_i^{n+\frac{1}{4}}\right)\right]$$

in which $\lambda_{half} = \frac{0.5\Delta t}{\Delta x}$ and the terms $\mathbf{u}_{i+1}^{n+\frac{1}{4}}$ and $\mathbf{u}_i^{n+\frac{1}{4}}$ are as follows:

$$\mathbf{u}_{i+1}^{n+\frac{1}{4}} = \mathbf{u}_{i+1}^n + \frac{\lambda_{half}}{2}d\mathbf{F}_{i+1} \quad (7)$$
$$\mathbf{u}_i^{n+\frac{1}{4}} = \mathbf{u}_i^n + \frac{\lambda_{half}}{2}d\mathbf{F}_i$$

Making use of the representation $\mathbf{F}_i = \mathbf{f}(\mathbf{u}_i^n)$ to approximate the partial derivatives scaled by Δx gives:

$$d\mathbf{F}_{i+1} = MM(\mathbf{F}_{i+2} - \mathbf{F}_{i+1}, \mathbf{F}_{i+1} - \mathbf{F}_i) \quad (8)$$
$$d\mathbf{F}_i = MM(\mathbf{F}_{i+1} - \mathbf{F}_i, \mathbf{F}_i - \mathbf{F}_{i-1})$$

Herein MM is the MinMod limiter [15] which can be defined for two scalar arguments a and b as:

$$MM(a,b) = \quad (9)$$
$$\frac{1}{2}\big(sign(a) + sign(b)\big)\min(|a|,|b|)$$

Likewise $d\mathbf{u}_i$ and $d\mathbf{u}_{i+1}$ in Eq. (6) are evaluated:

$$d\mathbf{u}_{i+1} = MM(\mathbf{u}_{i+2} - \mathbf{u}_{i+1}, \mathbf{u}_{i+1} - \mathbf{u}_i) \quad (10)$$
$$d\mathbf{u}_i = MM(\mathbf{u}_{i+1} - \mathbf{u}_i, \mathbf{u}_i - \mathbf{u}_{i-1})$$

Eq. (6) provides a piecewise constant approximation at time level $n + \frac{1}{2}$. The second step is exactly the same as Eq. (6) with some changes:

$$\mathbf{u}_i^{n+1} = \frac{1}{2}\left(\mathbf{u}_{i-\frac{1}{2}}^{n+\frac{1}{2}} + \mathbf{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}}\right) + \quad (11)$$
$$\frac{1}{8}\left(d\mathbf{u}_{i-\frac{1}{2}} - d\mathbf{u}_{i+\frac{1}{2}}\right) +$$
$$\lambda_{half}\left[\mathbf{f}\left(\mathbf{u}_{i+\frac{1}{2}}^{n+\frac{3}{4}}\right) - \mathbf{f}\left(\mathbf{u}_{i-\frac{1}{2}}^{n+\frac{3}{4}}\right)\right]$$

The variables of Eq. (11) provide an analogous relation to Eqs. (7, 8and10) if therein substitute indices $n + \frac{1}{4}$ to $+\frac{3}{4}$, n to $n + \frac{1}{2}$, i to $i - \frac{1}{2}$, $i + 1$ to $i + \frac{1}{2}$, $i - 1$ to $i - \frac{3}{2}$ and $i + 2$ to $i + \frac{3}{2}$.

2.5. Lax-Wendroff and Lax-Wendroff with nonlinear filter methods

The Lax-Wendroff finite-difference scheme can be derived from a Taylor-series expansion [12]. This scheme is second-order accurate with truncation error order of $O[(\Delta x)^2, (\Delta t)^2]$. It is also stable for $\frac{c\Delta t}{\Delta x} \leq 1$. When the scheme is applied to Eq. (3), an explicit two-step three-time-level discretization is derived [15]. The first time step of LxW is identical to the first time step of LxF method (Eq. 4). The other step is:

$$\frac{\mathbf{u}_i^{n+1} - \mathbf{u}_i^n}{\Delta t} = \frac{\mathbf{f}(\mathbf{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}}) - \mathbf{f}(\mathbf{u}_{i-\frac{1}{2}}^{n+\frac{1}{2}})}{\Delta x} \rightarrow \quad (12)$$
$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n + \frac{\Delta t}{\Delta x}(\mathbf{f}(\mathbf{u}_{i+\frac{1}{2}}^{n+\frac{1}{2}}) - \mathbf{f}(\mathbf{u}_{i-\frac{1}{2}}^{n+\frac{1}{2}}))$$

Actually, step one is the LxW method applied at spatial midpoints $(i + \frac{1}{2})$ and at half time increments, and step two invokes the leapfrog procedure for the remaining half time increment. If the result of step one is substituted into step two, the original Lax-Wendroff procedure is given [12]. The stencil of the described two-step LxW is plotted in Fig. 3. LxW method is dispersive so one may use an option to follow each step with a nonlinear filter to reduce the total variation of the numerical solution [13]. The method is then called smoothed Lax-Wendroff (SLxW). The aforementioned filter is usually defined as follows [15]:

$Dp = \mathbf{u}_{i+1}^{n+1} - \mathbf{u}_i^{n+1}, Dm = \mathbf{u}_i^{n+1} - \mathbf{u}_{i-1}^{n+1}$ (13)

If the multiplication of *Dp* and *Dm* is negative, the nonlinear filter is applied and the results of LxW are amended as follows (SLxW). Otherwise, the results do not change.

$$if \quad |Dp| > |Dm| \rightarrow \qquad\qquad (14)$$

$$\begin{cases} \mathbf{u}_j^{n+1} = \mathbf{u}_j^{n+1} + corr(Dp, Dm) \\ \mathbf{u}_{j+1}^{n+1} = \mathbf{u}_{j+1}^{n+1} - corr(Dp, Dm) \end{cases}$$

$$if \quad |Dp| \le |Dm| \rightarrow$$

$$\begin{cases} \mathbf{u}_j^{n+1} = \mathbf{u}_j^{n+1} + corr(Dp, Dm) \\ \\ \mathbf{u}_{j-1}^{n+1} = \mathbf{u}_{j-1}^{n+1} - corr(Dp, Dm) \end{cases}$$

$$if \quad |Dp| > |Dm| \rightarrow \qquad\qquad (15)$$

$$corr(Dp, Dm) = sign(Dp) \times \min(|Dm|, \tfrac{|Dp|}{2})$$

$$if \quad |Dp| \le |Dm| \rightarrow$$

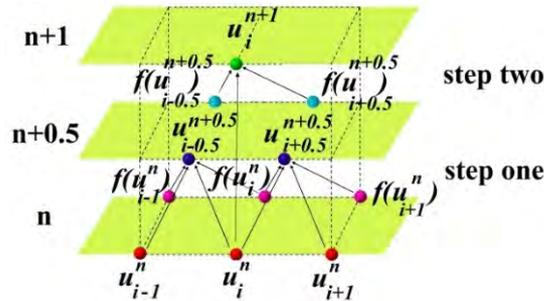$$corr(Dp, Dm) = sign(Dp) \times \min(|Dp|, \tfrac{|Dm|}{2})$$

Fig. 3. Stencil of two-time steps LxW.

2.6. Initial and boundary conditions

The initial conditions are taken according to the steady state situation of the system. The boundary conditions describe the situation at the ends of the pipeline, e.g. reservoir, junction, pump or valve [16]. Since the mesh is staggered herein, there is no need to construct approximate solutions on the boundaries at the first half step for either method.

In this study, the pipe system consists of a reservoir at the upstream end of the pipeline and

a valve at the downstream end discharging to the atmosphere (Fig. 4**)**. Two boundary conditions are required at the two ends i.e. at *x* = 0 and at *x = L*. For the reservoir at the upstream end of a piping system (*x*=0) a constant pressure is prescribed $H_1^{n+1} = H_0$, in which subscript "0" shows the steady state situation of the system. This relation can be combined with the discretized form of Eq. (2) at the first point of the region (reservoir) to directly obtain its velocity variations:

$$\frac{V_1^{n+1} - V_1^n}{\Delta t} = -g \frac{H_2^n - H_1^n}{\Delta x} \rightarrow \qquad (16)$$
$$V_1^{n+1} = V_1^n + \frac{-g\Delta t}{\Delta x}(H_2^n - H_1^n)$$

For a downstream valve at *x=L* with instantaneous closure V_M^{n+1}=0, in which subscript "M" refers to the valve computational section. Eq. (1) can now be discretized at the last point of the region (valve) to arrive at pressure head variations at the valve:

$$\frac{H_M^{n+1} - H_M^n}{\Delta t} = -\frac{c^2}{g} \frac{V_M^n - V_{M-1}^n}{\Delta x} \qquad (17)$$
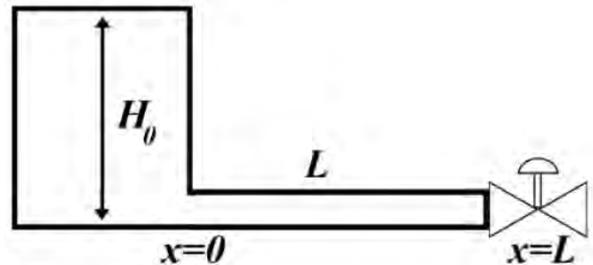$$\rightarrow \; H_M^{n+1} = H_M^n + \frac{-c^2}{g} \frac{\Delta t}{\Delta x}(V_M^n - V_{M-1}^n)$$

Fig. 4. Reservoir-pipe-valve system.

For a valve with non-instantaneous closure, the following relation between pressure head H_M^{n+1} and velocity V_M^{n+1} holds [7]:

$$V_M^{n+1} = \frac{V_0 \tau}{\sqrt{H_0}} \sqrt{H_M^{n+1}}, \;\; \tau = \frac{(c_d A_v)^{n+1}}{(c_d A_v)_0}$$
$$\qquad (18)$$

where H_0, V_0 are steady state head and velocity

at the upstream end of the valve respectively. The opening ratio of the valve, τ, is usually defined by the manufacturer over time. It is a function of c_d = discharge coefficient and A_v = opening area of the valve which depends on its type and function. In the current simulation, the following function for $\tau(t)$ which is specified based on measurements ball-valve is used [7].

$$\tau(t) = \begin{cases} \left(1 - \frac{t}{T_c}\right)^{3.53} & for \quad 0 \le t \le 0.4T_c \\ 0.394\left(1 - \frac{t}{T_c}\right)^{1.70} & for \quad 0.4T_c \le t \le T_c \\ 0 & for \quad t \ge T_c \end{cases}$$

$$(19)$$

in which T_c is the duration of the valve closure. Equations (17 and 18) can be combined to provide a direct relation for the velocity computation at the valve point:

$$V_M^{n+1} = \frac{V_0\tau}{\sqrt{H_0}}\sqrt{H_M^n + \frac{-c^2}{g}\frac{\Delta t}{\Delta x}(V_M^n - V_{M-1}^n)} \quad (20)$$

3. Verification of numerical model

The numerical solutions presented in the previous section are implemented in MATLAB codes. To validate the developed computer codes, a test problem is taken into account. The problem defines instantaneous and gradual valve closure in the reservoir-pipe-valve system shown in Fig. 4. The properties of the test problem are: length of pipe = 50 m, diameter of pipe = 0.2 m, pressure wave speed = 1195.2 m/s, steady state velocity = 0.4 m/s, reservoir head = 10 m, valve closure time = 0.029 s. The results of the proposed methods are then compared with MOC and Godunov solutions.

In Fig. 5, the head time history of the valve is compared with corresponding results of MOC and Godunov schemes for instantaneous closure. As seen, the head computations of LxF and NT are in a good agreement with those of MOC and Godunov. The LxW and SLxW simulate water hammer with a lot of fluctuations at discontinuities.

The same set of results but here for gradual valve closure is plotted in Fig. 6. As they manifest, in the case of gradual valve closure, the maximum relative error of LxW and SLxW remarkably decreases compare to MOC and

Godunov. In addition, all spurious head fluctuations in discontinuities are eliminated in comparison with instantaneous valve closure. On the whole, the four proposed methods provide satisfactory results especially when transients caused by gradual valve closure.
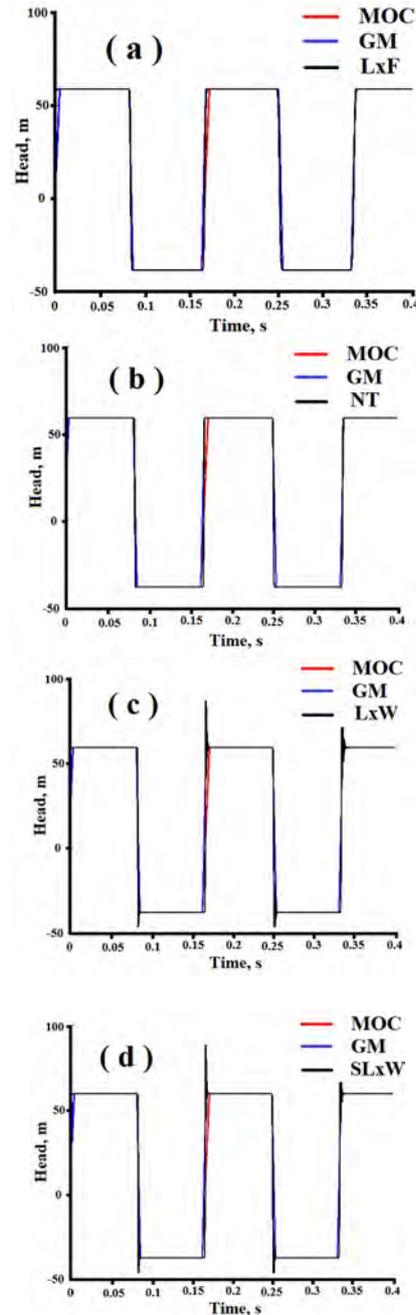

Fig. 5. Pressure head comparison of the instantaneous closure of the valve for simulations using MOC and Godunov (GM) schemes with (a) LxF (b) NT (c) LxW (d) SLxW.

Table 1 compares the maximum relative error of LxF, NT, LxW and SLxW schemes with MOC and Godunov. The error is evaluated as follows:

Relative error =

$$\frac{\text{calculations with (proposed methods} - \text{MOC or Godunov)}}{\text{calculations with MOC or Godunov}} \quad (21)$$

The adapted nonlinear filter for the SLxW causes to less relative error than the LxW. The maximum relative error of LxF and NT are very small so as to make appropriate solutions together with MOC and Godunov.

Table 1. Comparison of the maximum relative error LxF, NT, LxW and SLxW with MOC and Godunov.

Method	compared MOC		compared Godunov	
	Instantaneous	Gradual	Instantaneous	Gradual
LXF	0.0023	0.1154	0.0425	0.1043
NT	0.00005	0.1087	0.00002	0.1139
LxW	0.1749	0.1195	0.1750	0.2199
SLxW	0.1330	0.1154	0.1331	0.2199

Another comparison is made in Table 2 which shows the run-time durations for all mentioned methods in two cases of instantaneous and gradual valve closure. The above-explained methods are all fast, and manifest much fewer run times in comparison to the Godunov scheme.

This is because Godunov method requires solving a Riemann problem for its time advancing yet none of the proposed explicit schemes are needed.

Table 2. Run times for instantaneous & gradual valve closure.

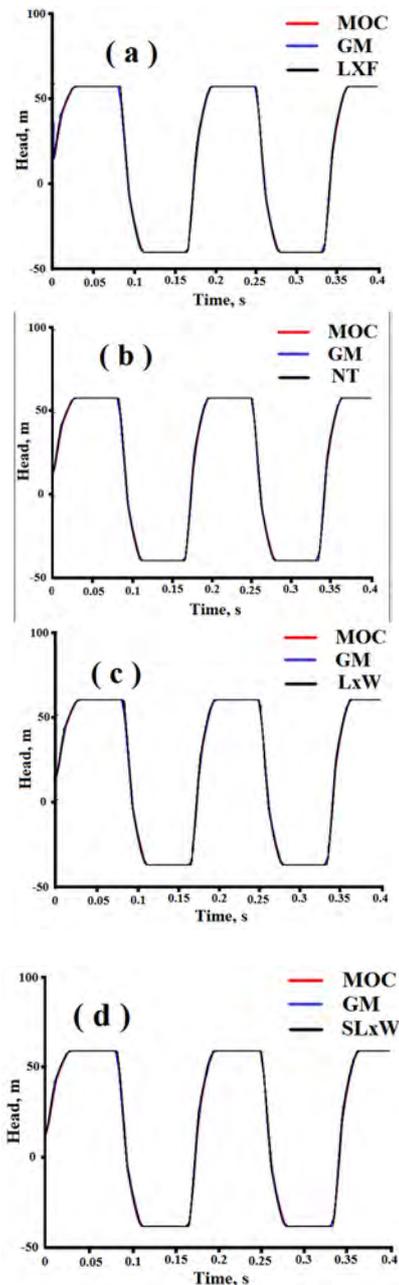| Method	Run times (s)	
	Instantaneous	Gradual
LxF	0.49	0.65
NT	2	3.1
LxW	0.53	0.7
SLxW	1.2	1.3
MOC	0.02	0.03
Godunov	0.64	0.82

Fig. 6. Pressure head comparison of the gradually closure of the valve for simulations using MOC and Godunov (GM) schemes with (a) LxF (b) NT (c) LxW (d) SLxW.

4. Conclusions

The frictionless water hammer equations are solved using four explicit central finite difference schemes: LxF, NT, LxW, and SLxW. The schemes are implemented in MATLAB

codes to observe strengths and weaknesses of each one in terms of their accuracy and run-time duration. MOC and Godunov solutions of water hammer are considered as reference results with which the proposed finite difference solutions are compared. Two common cases of valve action being instantaneous and non-instantaneous closure, are studied.

The time history results of the pressure head revealed that LxW and SLxW simulate water hammer with many spurious fluctuations at times adjacent to discontinuities. Conversely, NT and LxF methods calculated pressures at all times even around discontinuities with an acceptable order of accuracy. Among the two methods, LxF method was less accurate than NT but it was much faster than NT. However, in the case of gradual closure, all of the proposed methods were in good agreement with those of MOC and Godunov.

The strengths of the proposed methods are the simplicity of the algorithms for numerical programming and the fast and accurate calculations. In addition, the independency of time and space steps allows for setting different spatial grid size with a unique time step. This, in turn, increases the accuracy of the method with respect to the conventional MOC and Godunov. It is therefore inferred that LxF and NT can be good alternatives for MOC and Godunov schemes as the latter methods often face with restrictions on selecting time or space steps.

5. Acknowledgments

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